Problem 1.29

Calculate the line integral of the function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$ from the origin to the point $(1, 1, 1)$ by three different routes:

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$.

(c) The direct straight line.

(d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

Solution

Part (a)

Use the fact that the integral is a linear operator to split it up over the line segments of the path.

$$\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} = \int_{\langle 0,0,0\rangle}^{\langle 1,0,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,0,0\rangle}^{\langle 1,1,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,1,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l}$$

Along the first line segment, the variation is solely over x while $y = 0$ and $z = 0$; along the second line segment, the variation is solely over y while $x = 1$ and $z = 0$; and along the third line segment, the variation is solely over z while $x = 1$ and $y = 1$.

$$\begin{aligned}
\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} &= \int_0^1 v_x\bigg|_{\substack{y=0\\z=0}} dx + \int_0^1 v_y\bigg|_{\substack{x=1\\z=0}} dy + \int_0^1 v_z\bigg|_{\substack{x=1\\y=1}} dz \\
&= \int_0^1 x^2\,dx + \int_0^1 2y(0)\,dy + \int_0^1 (1)^2\,dz \\
&= \frac{1}{3} + 0 + 1 \\
&= \frac{4}{3}
\end{aligned}$$

Part (b)

Use the fact that the integral is a linear operator to split it up over the line segments of the path.

$$\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} = \int_{\langle 0,0,0\rangle}^{\langle 0,0,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,0,1\rangle}^{\langle 0,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,1,1\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l}$$

Along the first line segment, the variation is solely over z while $x = 0$ and $y = 0$; along the second line segment, the variation is solely over y while $x = 0$ and $z = 1$; and along the third line segment, the variation is solely over x while $y = 1$ and $z = 1$.

$$\begin{aligned}\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} &= \int_0^1 v_z\bigg|_{\substack{x=0\\y=0}} dz + \int_0^1 v_y\bigg|_{\substack{x=0\\z=1}} dy + \int_0^1 v_x\bigg|_{\substack{y=1\\z=1}} dx \\ &= \int_0^1 (0)^2\,dz + \int_0^1 2y(1)\,dy + \int_0^1 x^2\,dx \\ &= 0 + 1 + \frac{1}{3} \\ &= \frac{4}{3}\end{aligned}$$

Part (c)

In order to do the line integral over the straight line from $\langle 0,0,0\rangle$ to $\langle 1,1,1\rangle$, parameterize this line: $\mathbf{l}(t) = \langle t,t,t\rangle$, where $0 \le t \le 1$.

$$\begin{aligned}\int_{\langle 0,0,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} &= \int_0^1 \mathbf{v}(\mathbf{l}(t))\cdot\mathbf{l}'(t)\,dt \\ &= \int_0^1 \langle t^2, 2t^2, t^2\rangle\cdot\langle 1,1,1\rangle\,dt \\ &= \int_0^1 (t^2 + 2t^2 + t^2)\,dt \\ &= 4\int_0^1 t^2\,dt \\ &= \frac{4}{3}\end{aligned}$$

Note that it doesn't matter what path is taken from $\langle 0,0,0\rangle$ to $\langle 1,1,1\rangle$. The line integral will always yield $4/3$ because $\mathbf{v}$ is conservative:

$$\begin{aligned}\nabla\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2yz & y^2\end{vmatrix} &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial z}(2yz)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial z}(x^2)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(x^2)\right] \\ &= \hat{\mathbf{x}}(2y - 2y) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) \\ &= \mathbf{0}.\end{aligned}$$

Part (d)

The line integral around the closed loop that goes out along path (a) and back along path (b) is

$$\begin{aligned}
\oint \mathbf{v}\cdot d\mathbf{l} &= \int_{\langle 0,0,0\rangle}^{\langle 1,0,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,0,0\rangle}^{\langle 1,1,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,1,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,1,1\rangle}^{\langle 0,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,1,1\rangle}^{\langle 0,0,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,0,1\rangle}^{\langle 0,0,0\rangle} \mathbf{v}\cdot d\mathbf{l} \\
&= \int_{\langle 0,0,0\rangle}^{\langle 1,0,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,0,0\rangle}^{\langle 1,1,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,1,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} - \int_{\langle 0,1,1\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} - \int_{\langle 0,0,1\rangle}^{\langle 0,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} - \int_{\langle 0,0,0\rangle}^{\langle 0,0,1\rangle} \mathbf{v}\cdot d\mathbf{l} \\
&= \left(\int_{\langle 0,0,0\rangle}^{\langle 1,0,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,0,0\rangle}^{\langle 1,1,0\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 1,1,0\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l}\right) - \left(\int_{\langle 0,0,0\rangle}^{\langle 0,0,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,0,1\rangle}^{\langle 0,1,1\rangle} \mathbf{v}\cdot d\mathbf{l} + \int_{\langle 0,1,1\rangle}^{\langle 1,1,1\rangle} \mathbf{v}\cdot d\mathbf{l}\right) \\
&= \left(\frac{4}{3}\right) - \left(\frac{4}{3}\right) \\
&= 0.
\end{aligned}$$